Numerical Solutions to Systems of Linear Equations

Dave Pawlowski

Phy 379

Today, we will investigate how to use a computer to solve a system of linear equations, i.e.:

$$ax + by + cz = \alpha$$
$$dx + ey + fz = \beta$$
$$gx + hy + iz = \gamma$$

where x, y, and z are the only unknown variables.

Given the appropriate values, you could solve this problem with little trouble. But what if instead of having 3 equations, there were 30?

To illustrate one of the techniques to be used to solve a system, let's start with an example system:

$$\begin{aligned} x + y + z &= 0 \\ x - 2y + 2z &= 4 \\ x + 2y - z &= 1 \end{aligned}$$

We can write this system as the matrix equation

$$A \cdot X = C$$

where A, X, and C are given by:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The matrix A is called the matrix coefficient of the linear system. We can write an **augmented matrix** associated with the system [A|C]:

$$[A|C] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & -1 \end{array}\right]$$

In general, if the linear system has ***n*** equations with ***m*** unknowns, then the matrix coefficient is an ***n x m*** matrix and the augmented matrix is ***n*** x ***m+1.***

By determining the augmented matrix, it is possible to solve the system numerically using iteration. There are number of ways to accomplish this, but one of the most widely used techniques is called **Gaussian Elimination.**

Gaussian elimination has two parts. The first step is reducing the augmented matrix to a triangular matrix or put in echelon form.

$$\mathbf{L} = \begin{bmatrix} l_{1,1} & & & & 0 \\ l_{2,1} & l_{2,2} & & & \\ l_{3,1} & l_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ l_{n,1} & l_{n,2} & \dots & l_{n,n-1} & l_{n,n} \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \dots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \dots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

The second step is to use back-substitution to find the solution (also referred to as reduced row echelon form).

The first step is accomplished through the use of **elementary row operations,** specifically, multiplying rows, switching rows, and adding multiples of rows to other rows.

In order to put the matrix in echelon form, we want all elements of the matrix below the **diagonal** to be zero.

$$[A|C] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & -1 \end{array}\right]$$

$$\mathbf{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \ldots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \ldots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

Starting on the left side on the matrix and moving right, the **pivot** column is the column that we want to transform as we work through the matrix. The pivot column starts as the first column on the left. The **pivot point** is the diagonal element in the pivot column.

$$[A|C] = \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & -1 \end{array}\right]$$

The pivot point determines the operation that we will use to reduce the matrix, as we step through it.

For each pivot column, the following steps are performed:

1) Locate the **pivot point**. The row containing the pivot point is the **pivot row**. Divide every element in the pivot row by the pivot to get a new pivot row with a 1 in the pivot position.

2) Subtract a multiple of the pivot row from each of the rows below it to get a 0 in each position in the pivot column below the pivot point.

The First Column

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & -1 \end{array}\right]$$

Pivot point

Pivot Column

Pivot Row

Step 1) Our pivot is already 1, so we don't have to divide

Step 2) Now for each row, we want a zero in the pivot column. So we subtract 1x the pivot row from the 2nd row:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 1 & 2 & -1 & -1 \end{array}\right]$$

and subtract 1x the pivot row from the 3nd row:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

The 2nd column

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

Pivot point

Pivot Column

Pivot Row

Step 1) This time, our pivot is -3, so we divide the row by that:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 1 & -2 & -1 \end{array}\right]$$

Step 2) Now subtract 1x the pivot row from 3rd row

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & -\frac{5}{3} & \frac{1}{3} \end{array}\right]$$

The matrix is now in **row echelon** form

At this point, you should start to see a solution to one of our equations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & -\frac{5}{3} & \frac{1}{3} \end{array}\right]$$

Remember that we are solving the equation A X = C, so this says that our original unknown, z = -1/5.

The next part of the Gaussian Elimination scheme is back substitution- which means that we take the variable that we just solved for and plug it in to one of our other equations.

We now have the following equation.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & -\frac{1}{5} \end{array}\right]$$

We want to **diagonalize** the matrix, so that there are zeros above and below the matrix (Doing this turns our method into Gauss-Jordan elimination). Note that the result of this is that we will turn our original coefficient matrix into the **identity** matrix.

To do this, we again use elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & -\frac{1}{5} \end{array}\right]$$

Multiply the 3rd row by -1/3 and subtract from row 2.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\frac{21}{15} \\ 0 & 0 & 1 & -\frac{1}{5} \end{array}\right]$$

Multiply the 3rd row by 1 and subtract from row 1.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & -\frac{21}{15} \\ 0 & 0 & 1 & -\frac{1}{5} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & -\frac{21}{15} \\ 0 & 0 & 1 & -\frac{1}{5} \end{array}\right]$$

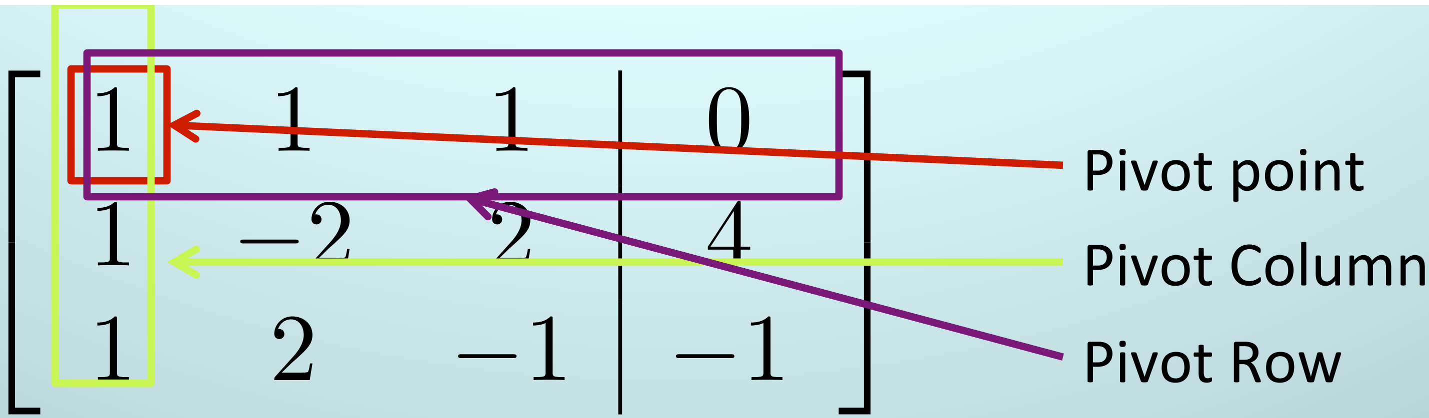
Now, multiply 1x the 2nd row and subtract from the 1st:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{24}{15} \\ 0 & 1 & 0 & -\frac{21}{15} \\ 0 & 0 & 1 & -\frac{1}{5} \end{array}\right]$$

And we get the solution:

$$x = \frac{8}{5}, y = -\frac{7}{5}, z = -\frac{1}{5}$$

While some of this should seem familiar from solving systems of equations by hand, it should seem a bit more complicated. This is so that we can actually write a code that works.



$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & -1 \end{array}\right]$$

A simple approach to putting the matrix in row echelon form is below. Note that this wont work for all systems. If the value at a pivot point is zero, then that value can not be used to pivot, in which case you would have to move the pivot point over on spot and continue.

```
M = matrix

For all pivots:
    M[ipivot,:] = M[ipivot,:]/M[ipivot,ipivot]

    for all rows that haven't been done yet:
        M[irow,:]=M[irow,:] - M[ipivot,:]
```